The Fractalopolis model - a sustainable approach for a central place system

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1 Introduction

Urban sprawl generates often diffuse settlement patterns where residential areas are localized far away from jobs as well as from retail centers and services. Hence periurbanization tends to increase the number and the length of trips and hence energy consumption and pollution. This negative impact has essentially been made evident by Newman and Kenworthy ([NR89]). Sustainable development requires better managing this dynamics what risks, moreover, undermining natural and agricultural areas. Numerous authors recommend coming back to compact cities in order to limit urban sprawl. However policies favoring the compact city turned out to be less efficient as expected. They induce an increase of the costs of housing, traffic congestion and reduce the accessibility of leisure areas ([Bre97]). A large number of households who choose these areas flee urban density since they prefer living in individual houses surrounded by a garden and enjoy a green and calm environment. Even if this lifestyle contributes to urban sprawl ([VHF02]), such households will reject densifying ([Bre97], [GR97], [Fou95]). As pointed out Schwanen et al. ([SD04]) households tend to minimize the distance or the travel time for acceeding to their jobs ([MD97]), but also to retail service centers ([Ler76]) or even leisure areas ([GB02]). Hence, in order to manage mobility, to reduce energy consumption and to prevent from fragmentation of build-up areas as well as of open landscape, we develop in the following a new planning concept. By improving accessibility to the different kinds of sites residents frequent more or less regularly, this concept aims to reduce the negative impacts of periurbanisation without rejecting it. Leisure areas of in the neighborhood of urbanized areas provide the desired quality of live for residents.

Several authors emphasize the importance of developing secondary centers or polycentric urban networks ([Fou95])what incites us to reflect about an planning apporoach which starts by reflcting on central place theory and which introduces a hierarchical system of cities according to the catchment areas of their public and private services. A critical reflection about the underlying spatial system leads us to introduce a multi-scale approach of planning which is inspired from fractal geometry ([Fra08]).

2 Towards a new central place model

The classic central place theory of Christaller ([Chr80]) introduces a system of cities consisting of different levels. The town belonging to a particluar level provide a certain type of public and private services. The system follows a hierarchical logic where few big cities offer high level services with large catchment areas including an important number of small and mean-sized cities. The lower level services are present in a huger number of cities whose catchment areas are smaller, finally very small cities ensure only supply in daily needs.

Globally the Christaller systems follows a "multiplicative logic" (geometrical series) for both the systems, the number of cities belonging to a certain supply level, and the mean population numbers of these cities. It is well-known that linking two geometrical series allows obtaining a hyperbolical distribution law relating in the present case the size of cities to their numbers of occurrence. This is in concordance with the rank-size distribution of cities, the so-called Zipf law ([D82], [F95]), but such relationships are often observed in economics, in nature and correspond also to fractal geometry ([Man87]).

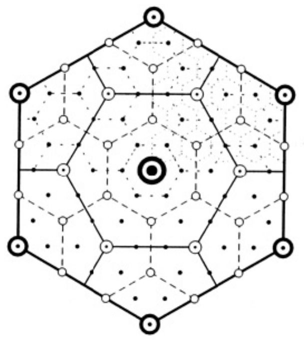

Figure 1: The Christaller system of cities (Christaller, 1933)

A particular feature of Christallers' system is the fact that cities are uniformly dsitributed in space (cf. figure 1). Arlinghaus ([Arl85]) introduces the same logic in her paper aiming to introduce a fractal approach of central place theory. Let us however remind that this papers generates just the borderlines of the catchment areas of the towns belonging to the different hierarchical levels by means of generateor resembling to that of teragons or Koch curves.

The development of a settlement system is closely linked to the transportation network ensuring accessibility. Covering territories uniformly by a transportation network is rather expensive and urban sprawl is just one of the consequence of providing good accessibility all over space. Hence we may ask the question if in the context of regional and urban planning it is really the good solution referring to such a uniform distribution principle for central places or if it would be more useful to concentrate activities in the vicinity of some transportation axes. Such ideas are not new. In Northern European metropoles such concept have been proposed for managing urban sparwl as shows the famous finger plan for Copenhagen, he palm-plan for Hamburg or the development strategies developped for Berlin favouring an extension along the suburban railway axes (fig. 2). More recently the idea of a "transport oriented development"

(TOD) refers to similar reflexions ([Cal93]).

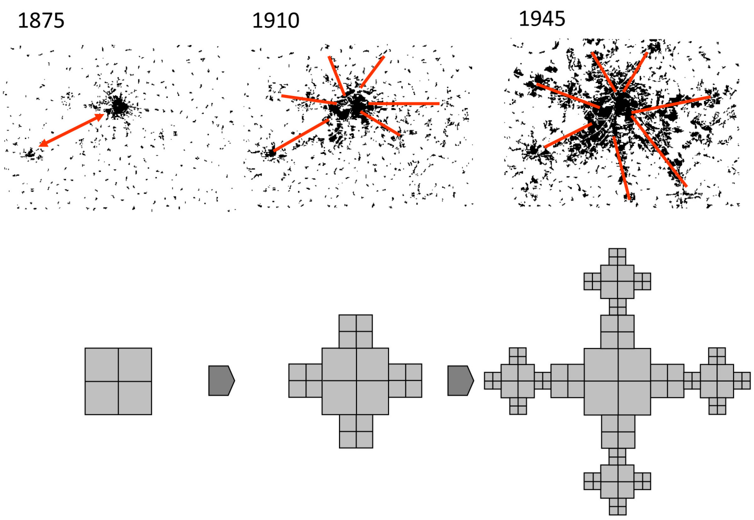

Figure 2: The urban growth of Berlin along suburban railway axes and the first steps for generating a multifractal Sierpinski carpet

However we should be aware that only the TOD and the Berlin concept introduce really subcenters in the hinterland of the main center and thus refer to a central place hierarchy. By distiguishing facilities according to their attendance rate such a hierarchy seems indeed of interest. Indeed we may accept longer distances for acceeding to rarely used facilities, but daily needs should benefit from a good accessibility in order to reduce taffic flows and hence pollution, but also consumption of space, since the construction of transportation axes on a regional level consumes much space. E.g. it has been shown that in the fringe of the greater Paris region between 1987 and 1997, 1,4% of space has been consumed for housing or mixed use buildings, wheras more then 50% has been used for constructing roads ([Tou06]).

This incited us to reflect about a planning concept which aims combining a central place hierarchy with a transport oriented logic. Hence the spatial organization of the proposed central place system follows strongly a hierarchical principle for the offer of facilities, similar to that of Christaller, but concentrates city development close to transportation axes which should rather be interpreted as public transportation network axes. In order to develop a spatial model the concept makes use of the scaling properties of fractal geoometry and refers more particularly to the logic of Sierpinski carpets[1]. The concept should be considered as *descriptive and normative*, serving as reference for planning purposes. We focus here on the theoretical framework, but the model can be applied to real world situation without loss of the basic principles. These principles have been implemented in a planning support system which provides GIS-facilities for developping and evaluating planning scenarios[2]. The basic principles based

[1]There use for describing urban paterns has been shown e.g. in [Fra94] or more recently in [TFB08]

[2]This software paquage has been developped by Gilles Vuidel, software engeneer at the

on a unifractal approach as well as concrete applications for planning have been presented up to now in several, mostly French or German written papers ([FTVH11], [Fra07],[Fra11], [FHTV07], [TVFH10]), [CYFVT11]).

The goal of the present working paper is to enlarge the model by using a multifractal logic (cf. also [CYF11]) and by focussing of the population distribution in the considered city system. But first we remind the the underlying logic of the concept which takes into account simultaneously different kinds of objectives:

- reducing of the travel length for acceeding to higher order facilities,
- respecting the diversity of social demand i.e. taking into account that certain types of households prefer living in a calm, low dense environment which allows good access fo green amenities,
- avoiding leapfrogging that lengthens the distances to acced to centers,
- avoiding the fragmentation of natural or agricultural areas.

For this aim the proposed spatial model uses iterative mapping procedures similar to those used for generating multifractal Sierpinski carpets. We assume that there exists a hierarchically structured system of central places according to different levels of services and commercial offer they provide. However, contrarily to Christaller, we will see later that towns belonging to the same hierarchical level have no longer the same population. Towns of a given level but lying close to a rather high ranked center are assumed to concentrate more population than those lying close to lower ranked centers.

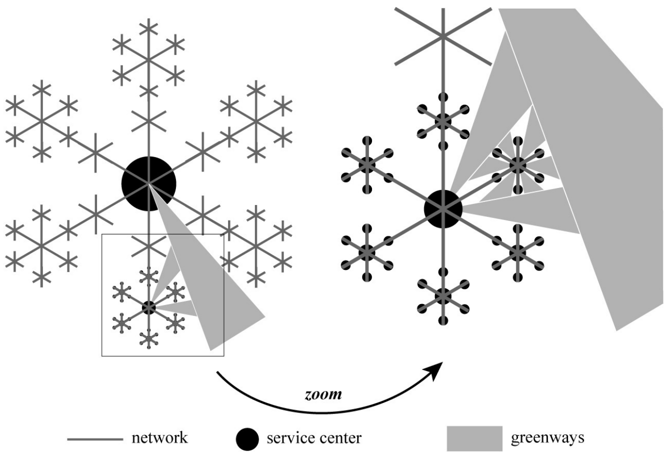

Figure 3: A modified Christaller scheme

Figure 3 shows a system which corresponds to our logic. This system reminds by his hexagonal shape the Christaller scheme. However we see that the towns are concentrated in the vicinity of axes which may be interpreted as public transportation axes. Between the axes exists another connected spatial system which corresponds to undeveloped areas and which we interpret as natural and

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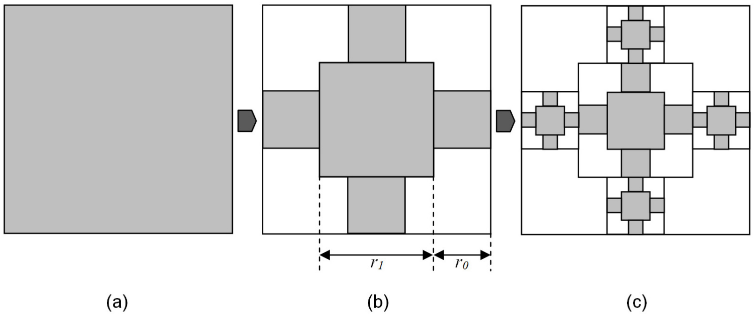


Figure 4: Generating stepwise a multifractal Sierpinski carpet

agricultural reserves including rural settlements. By its shape this system avoids obviously a fragmentation of these rural and natural zones. On the other hand the peripheric branches of the transportation network are *not* linked. This helps reducing traffic flows between supply centers belonging to the same hierarchical level.

In order to introduce a quantitative modeling approach in a convenient way, we consider now a simpler model version which follows the same logic (cf. figure 4). Let us start by drawing a large sized square of a certain base length which we normalize to one (figure 4(a). We assume that the most important center of our system is localized in the centroid of the square. The surface (in grey) of the square represents in some sense the catchment (or attraction) area of our central place. We now introduce a generator, represented in figure 4(b) consisting of one central square of base length $r_1 < 1$ centered on the previously introduced first order center dotted in the figure. This square is surrounded by $N = 4$ smaller ones with base length $r_0 < r_1$. Let us emphasize that the generator lies just within the initial square so that the outer corners of peripheral squares are identical to that of the initial square. Moreover no overlapping of squares is allowed. We assume now that this first step corresponds to the implementation of $N = 4$ second order centers localized in the centroids of the smaller squares. The surface of the squares correspond now to the direct catchment areas of these centers, or more precisely, they define the areas wherein we favour future development. This means that we assume that distances to the center from settlements lying within this area are acceptable. According to its level the central square has a bigger second level catchment area than the peripheral centers. In the next step we reiterate the procedure shown in figure 4 b. Each of the existing squares is then replaced by a smaller replication of the generator. According to our logic, we conserve of course the already generated first order and second order central places and we add third order central places lying within the direct catchment areas of the second order centers. Again these centers are localized in the centroids od the generated smaller squares. By the iteration process the reduction factors r_1 and r_0 are combined according to all possible combinations what yields e.g. for the second step:

$$r_1 \cdot r_1, r_1 \cdot r_0, r_0 \cdot r_1, r_0 \cdot r_0$$

Of course since permutations are allowed we have

$$r_0 \cdot r_1 = r_1 \cdot r_0$$

This is the reason why the catchment area of the second order centers is the same then that of the third order centers belonging to the highest ranked center. This corresponds to a particularity of multifractal structures and we will come back to this topic when considering population distribution.

Another consequence of this feature of multifractals is that the direct catchment areas belonging to the third order centers have no longer the same size. We have small squares of base length $r_0 \cdot r_0$ and larger ones with base length $r_0 \cdot r_1$.

The next step adds another hierarchical level and we discover again that the size of the catchment areas of centers issued from different iteration steps and thus corresponding to different hierarchical levels is the same, and on the other hand that the catchment areas of centers belonging to the same level are different. Two logics can thus be distinguished:

- the first one generates the central place hierarchy by adding a lower level at each iteration step. Hence the iteration step where the centroid has been generated, determines its service level in the central place hierarchy.
- the second one is linked to the mentioned “degeneration” effect. Since permutations are allowed we have direct catchment areas which have the same size but belong to different service levels.

We should however emphasize that the logic of the spatial configuration of the centers corresponds to the logic of the central place theory. The fact that the area affected to the centers are of different size according to their localization seems an interesting feature since we can assume that cities lying closer to important high level centers are usually bigger than those lying close to low level centers. This logic will be reconsidered when defining the theoretical population numbers.

By going on with iteration, it is of course possible to generate a more hierarchical spatial system. Let us just remind that Christaller, e.g., distinguishes 7 different service levels. However in order to conserve a certain legibility, we restrict ourselves here to the four levels already introduced. These levels may be associated to the following kinds of attendance rates:

- level 1: rarely frequented services or shops
- level 2: monthly frequented services or shops
- level 3: weekly frequented services or shops
- level 4: daily frequented services or shops

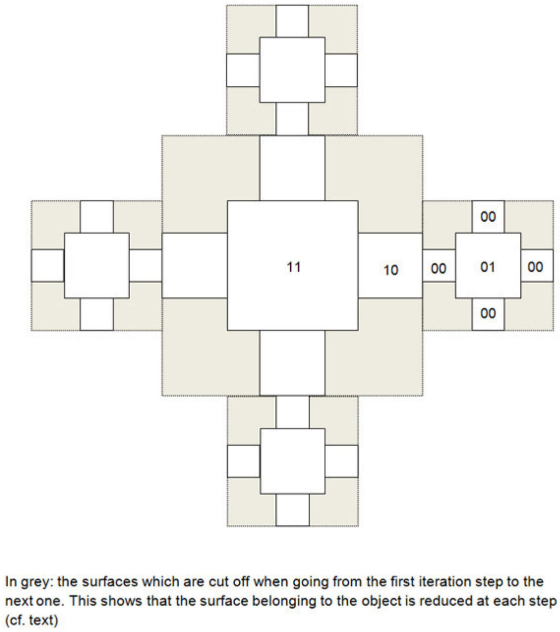
3 The coding system

We now introduce a coding system which allows distinguishing the different centers according to their service level. Hence for the first iteration step we distinguish the large central square which we denote by the digit 1 and the four smaller peripheral squares denoted by 0. In each following step we add now on the right of each digit another one, according the same logic. This makes evident that the hierarchy is created just by combining two factors. Hence in the next step the highest order central square is now called 11, the four adjacent generated smaller ones 10. The four peripheral squares generated in the previous step are replaced, too, by the generator. The occurring central place are called 01 and the four peripheral ones 00 (figure 5(a). This procedure is reiterated in the third step (cf.figure 5(b)). We obtain then a set of 8 different codes, each one consisting of three digits. The first level center with the highest facility level $m = 1$ has the code 111. The four directly adjacent squares of level $m = 2$ have the codes 110. They correspond to suburban areas of the main center. The four centers 011 correspond to the four centers of level $m = 2$ generated at the first iteration step. The peripheral centers centers 101 and 001 are issued from the second iteration step and correspond to centers of the facility level $m = 3$. Of course the 101 centers belong to the catchment area of 111 for higher level facilities, whereas the centers 001 belong to the catchment area of the second level centers 011. The small elements 100 and 000, adjacent to these third level centers, are all low level centers $m = 4$ (cf. figure 5(b)).

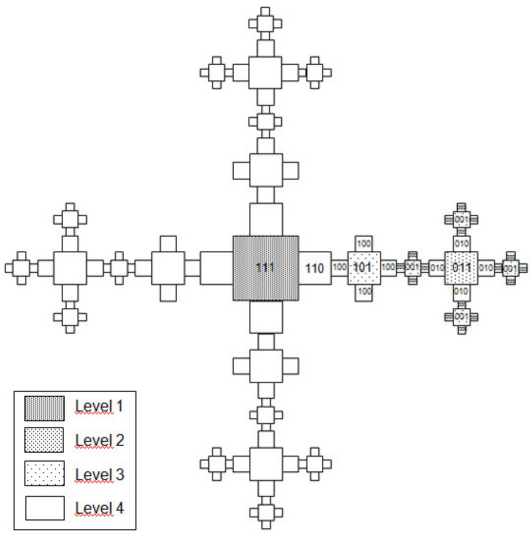
The stepwise generation of the elements can hence be represented as follows:

$$
\begin{array}{ccccccc}
 & & & & & & 111 \\
 & & & & & \nearrow & \\
 & & & & 11 & & \\
 & & & & & \searrow & \\
 & & & \nearrow & & & 110 \\
 & & 1 & & & & \\
 & & & \searrow & & & 101 \\
 & & & & & \nearrow & \\
 & \nearrow & & & 10 & & \\
 & & & & & \searrow & \\
 & & & & & & 100 \\
i & & & & & & \\
 & & & & & & 011 \\
 & & & & & \nearrow & \\
 & \searrow & & & 01 & & \\
 & & & & & \searrow & \\
 & & & \nearrow & & & 010 \\
 & & 0 & & & & \\
 & & & \searrow & & & 001 \\
 & & & & & \nearrow & \\
 & & & & 00 & & \\
 & & & & & \searrow & \\
 & & & & & & 000
\end{array}
$$

We can identify the following properties for the different kinds of elements :



(a)



(b)

Figure 5: Illsutration of the coding sytem. The figure (a) shows what ares are cut off from the urban system when going fom the generator to the nexrt iteration step. Figure (b) shows the generated central place hierarchy (cf. text)

code	level	next superior center	number	surface
111	1		1	$(S_1)^3$
110	4	1	4	$(S_1)^2 \cdot (S_0)$
101	3	1	4	$(S_1)^2 \cdot (S_0)$
100	4	3	16	$(S_1) \cdot (S_0)^2$
011	2	1	4	$(S_1)^2 \cdot (S_0)$
010	4	2	16	$(S_1) \cdot (S_0)^2$
001	3	2	16	$(S_1) \cdot (S_0)^2$
000	4	3	64	$(S_0)^3$

where we have set the basic surfaces as $(r_1)^2 = S_1$ and $(r_0)^2 = S_0$.

The codes inform us directly about the facility levels. Using a generalized code ijk, who obtain

$$\begin{array}{rcl} k = 0 & \Rightarrow & m = 4 \\ jk = 01 & \Rightarrow & m = 3 \\ ijk = 011 & \Rightarrow & m = 2 \\ ijk = 111 & \Rightarrow & m = 1 \end{array}$$

By introducing these codes we have given up the previously discussed commutativity. Indeed in the introduced system the codes 101 and 110 or 011 are not equivalent, even if the surface of their direct catchment area is the same. Hence the code introduces a non-commutative operation.

This has the consequence that the system shows some properties which correspond rather to properties of unifractals than to mulifractals, what is interesting in the given context. Hence, making abstraction of their size, we verify that the total number of centers belonging to the different levels follows a geometrical series, excepted the passage from the highest to the next level:

level	number	multiplicator
1	1	
2	4	4
3	20	5
4	100	5

This corresponds to the usual hierarchical logic observed in fractal structures.

4 The population model

We now focus on the population numbers what needs some preliminary reflections. We assume that the population model affects to the introduced catchment areas a certain amount of population. But first we should remind some fundamental features of multifractal geometry. Two basically different iteration concept must be distinguished :

- the first one used before resembles to the usual iteration procedure used e.g. for generating Sierpinski carpets and corresponds to the procedure we used before for generating our central place hierarchy. The procedure reduces an initially given figure subsequently by using several - in the present case two - reduction factors. As pointed out before, this corresponds to generate progressively a set of points corresponding to the centroids of the smaller copies of the initial figure. Hence the total areas belonging to the prefractal set - in our example the direct catchment areas of the centers of different order - is reduced at each iteration step.

- another way of proceeding is to give a certain mass, e.g. a population, and to dispatch this mass progressively according to an iterative mapping procedure on different parts of space. E. g. we may cover a given area by a grid consisting of large meshes and distribute the mass according to some weighting factors $p_0, p_1 \ldots$ among these different meshes. Then the iteration procedure generates within each mesh smaller ones and distributes the mass within this meshes according to the previously introduced distribution procedure. By this procedure the weighting factors are again combined in the same way as the previously introduced reduction factors, i.e. we obtain combined factors like $p_0^2, p_0 \cdot p_1, p_1^2$. This procedure is reiterated. However since we are reasoning in parts, the sum of mass over all grid elements remains constant over iteration, whereas the mass is more and more concentrated in the meshes which combines the high values of weighting factors.

In our case the situation is peculiar, since we combine a surface model with a weighting logic rather reminding the second approach. Let us remind that the first one serves generating a subset of areas which we consider as suitable for further urbanization. The second corresponds rather to o the distribution of population in the areas. However this does not really hold since, at each iteration step, we reduce the total amount of surface for which we assume future development as possible. This means that we subsequently put out of the system settlements, and thus population. This must be taken into account in the model.

Hence we propose the following model. We assume that when introducing the generator in the first step we split the population p living in the square-like initially selected area in two parts:

$$\begin{aligned} p &= \alpha \cdot p + (1-\alpha) \cdot p & (1) \\ &= \alpha \cdot p + p_{rur(1)} & (2) \end{aligned}$$

The population $p_{rur(1)}$ is the amount of the population living in the zones cut away by the generator and $(1-\alpha)$ is the corresponding part of the population. This makes evident that α is directly given by empirical data by the relation:

$$\alpha = \frac{p - p_{rur(1)}}{p}$$

The amount $\alpha \cdot p$ of the population lives in the "urban system" as it defined at this first iteration step. Hence we are reasoning in parts of population what becomes obvious by dividing (1) by p:

$$1 = \alpha + (1 - \alpha) \tag{3}$$

The urban system consists of the generator, i.e. of the center with code 1 and the four subcenters coded by "0". We assume now that this urban population is distributed among these 5 elements so that the main center concentrates the part a_1 of the urban population and each of the subcenters a_0. Hence we obtain:

$$\begin{aligned} p &= \alpha(a_1 + 4a_0)p + p_{rur(1)} \\ &= \alpha(a_1 + 4\frac{1-a_1}{4})p + p_{rur(1)} \end{aligned} \tag{4}$$

The second equation holds since we are reasoning in parts, i.e. $(a_1+4a_0) = 1$.

Hence the model affects the following population numbers to the different elements of the prefractal:

code	level	population	surface	density
1	1	$\alpha p a_1$	(S_1)	$\frac{\alpha p a_1}{(S_1)}$
0	2	$\alpha p a_0$	(S_0)	$\frac{\alpha p a_0}{(S_0)}$

Since α is already determined we have to compute a_1 and a_0 respecting the normalization requirement. Again we compute both the parameters directly by referring to empirical data distinguishing the center "1" from the four subcenters "0". If we call $\hat{p}_1$ the population living in the area "1" and $\hat{p}_0^{(i)}$ with $i = 1 \dots 4$ the empirical populations living in the surrounding subcenters, we obtain

$$\begin{aligned} a_1 &= \frac{\hat{p}_1}{\alpha p} \\ a_0 &= \frac{1}{4\alpha p}\sum_i \hat{p}_0^{(i)} \end{aligned}$$

Of course the normalization requirement is strictly fulfilled.

We now go on with iteration. In the next step we assume again that an amount of population $p_{rur(2)}$ lives in the parts of space now cut away by iteration (this ares is represnetd in grey in figure 5(a). We obtain the following relation:

$$p = \alpha\beta \cdot p + \alpha(1-\beta) \cdot p + p_{rur(1)} \tag{5}$$

$$\quad = \alpha\beta \cdot p + p_{rur(2)} + p_{rur(1)} \tag{6}$$

Again β is easy to compute according to the relationship:

$$\beta = \frac{p - p_{rur(2)} - p_{rur(1)}}{\alpha \cdot p}$$

This means that we estimate the parameters - here α, β etc. stepwise. Hence each iteration is consistent with itself - sums over the ratios of population affected to the different areas are by definition identical to the initally given total population. This seems strictly coherent with the iteration logic which can be stoppped at an arbitray iteration step what should not affect parameter values determined for previous steps. We vapply the same logic for all further introduced parameters.

We now must reflect about how distributing the population living within the new "urban system" among the elements belonging to our multifractal. If we would, according to the iterative logic of fractal geometry, use the logic we applied for constructing the multifractal Sierpinski carpet, we would obtain the following relation corresponding to (4):

$$\begin{aligned} p &= \alpha\beta(a_1(a_1 + 4 \cdot a_0) + 4a_0(a_1 + 4 \cdot a_0))p + p_{rur(2)} + p_{rur(1)} && (7) \\ &= \alpha\beta(a_1^2 + 4a_1 \cdot a_0 + 4a_0 \cdot a_1 + 4^2 a_0^2)p + p_{rur(2)} + p_{rur(1)} && (8) \end{aligned}$$

Again we would have dispatched the population according to the parts a_1 and a_0 among the different elements of the multifractal Sierpinski carpet. Since $a_1 \cdot a_0 = a_0 \cdot a_1$, the population would be the same for the cities with code 10, correponding to peripheric zones of the most important center and 01 what is the second order main center. By going on with iteration at the next step a center 110 with facility level 4 would have the same population as the level 3 center 101 or the level 2 center 011. Such a logic seems unrealistic since we should expect that e.g. the density in the level 2 center 011 is superior to that of the level 3 center center 101.

It is evident that this is a consequence of the commutative logic of multifractal iteration[3].

Hence we modify the strict iteration logic by allowing other ratios for the distribution of population in the subsequent iteration steps. Thus by introducing parts b_1 and b_0 for the second iteration step, we may rewrite the relation (7):

$$\begin{aligned} p &= \alpha\beta(a_1(b_1 + 4 \cdot b_0) + 4a_0(b_1 + 4 \cdot b_0))p + p_{rur(2)} + p_{rur(1)} \\ &= \alpha\beta(a_1 \cdot b_1 + 4a_1 \cdot b_0 + 4a_0 \cdot b_1 + 4^2 a_0 \cdot b_0)p + p_{rur(2)} + p_{rur(1)} \end{aligned}$$

Of course this assumption destroys the commutative logic and we clearly may distiguish centers of type 10 from those of type 01 with respect to there population numbers what holds, too, for the next step if we introduce additional factors c_1 and c_0.

It is evident that b_1 and b_0 are again linked by the requirement of normalisation, so that:

$$b_0 = \frac{1 - b_1}{4}$$

what yields for (9)

$$p = \alpha\beta \left(a_1 \left(b_1 + 4 \cdot \frac{1 - b_1}{4} \right) + 4 \cdot \frac{1 - a_1}{4} \left(b_1 + 4 \cdot \frac{1 - b_1}{4} \right) \right) p + p_{rur(2)} + p_{rur(1)} \quad (9)$$

[3]This is why the distribution function of the surfaces follows not a Pareto-distribution as unifractals do, but a binomial distribution, cf. [Fed88]

Hence we obtain again the model values of the population affected to the different elements generated at this iteration step:

code	level	population	surface	density
11	1	$\alpha\beta p a_1 b_1$	$(S_1)^2$	$\frac{\alpha\beta p a_1 b_1}{(S_1)^2}$
10	3	$\alpha\beta p a_1 b_0$	$(S_1)\cdot(S_0)$	$\frac{\alpha\beta p a_1 b_0}{(S_1)\cdot(S_0)}$
01	2	$\alpha\beta p a_0 b_1$	$(S_1)\cdot(S_0)$	$\frac{\alpha\beta p a_0 b_1}{(S_1)\cdot(S_0)}$
00	3	$\alpha\beta p a_0 b_0$	$(S_0)^2$	$\frac{\alpha\beta p a_0 b_0}{(S_0)^2}$

For determining the parameters b_1 and b_0 we proceed according to the previously defined logic which respects the different iteration steps. Hence α, β, a_1,a_0 are known. If we introduce the empirical data according to the same logic as previously, we obtain the empirical equation which is the equivalent to to (9):

$$p - p_{rur(2)} - p_{rur(1)} = \hat{p}_{11} + \sum_{i=1}^{4} \hat{p}_{10}^{(i)} + \sum_{i=1}^{4} \hat{p}_{01}^{(i)} + \sum_{i=1}^{16} \hat{p}_{00}^{(i)} \tag{10}$$

where we introduced empirical population numbers by a "hat" and we just restricted ourself to consider the only "urban population". We rewrite this equation and (9) in order to group the data referring to b_0 and those referring to b_1:

$$\begin{aligned} p - p_{rur(2)} - p_{rur(1)} &= \hat{p}_{11} + \sum_{i=1}^{4} \hat{p}_{01}^{(i)} + \sum_{i=1}^{4} \hat{p}_{10}^{(i)} + \sum_{i=1}^{16} \hat{p}_{00}^{(i)} \\ p - p_{rur(2)} - p_{rur(1)} &= \alpha\beta(a_1 \cdot b_1 + 4a_0 \cdot b_1 + 4a_1 \cdot b_0 + 4^2 a_0 \cdot b_0)p \end{aligned}$$

We now require that the part of the urban system which contains the parameter b_1 correponds to the total real population of this part what yields:

$$\begin{aligned} \hat{p}_{11} + \sum_{i=1}^{4} \hat{p}_{01}^{(i)} &= \alpha\beta(a_1 \cdot b_1 + 4a_0 \cdot b_1)p \\ &= \alpha\beta p \cdot b_1 \end{aligned} \tag{11}$$

where we took into account the normalisation $a_1 + 4a_0$. The same is required for b_0 what yields:

$$\begin{aligned} \sum_{i=1}^{4} \hat{p}_{10}^{(i)} + \sum_{i=1}^{16} \hat{p}_{00}^{(i)} &= \alpha\beta(4a_1 \cdot b_0 + 4^2 a_0 \cdot b_0)p \\ &= \alpha\beta p \cdot 4b_0 \end{aligned} \tag{12}$$

Both the relations are of course coherent, there sum is by definition the empirical population and the normalization $b_1 + 4b_0$ is verified. Relationships

(11) and (12), respectively the normalization, allow determining the parameters b_1 and b_0:

$$b_1 = \frac{\hat{p}_{11} + \sum_{i=1}^{4} \hat{p}_{01}^{(i)}}{\alpha\beta p} \tag{13}$$

$$b_0 = \frac{1 - b_1}{4} \tag{14}$$

In the same sense we now can go on with iteration. We then introduce a ratio γ of the population now affected to the elements belonging now to the urban system. Again normalized ratios c_1 and c_0 are introduced according to $c_1 + 4 \cdot c_0$. We restrict the discussion just by giving the populaton numbers as they are generated by the model:

code	level	population	surface	density
111	1	$\alpha\beta\gamma p a_1 b_1 c_1$	$(S_1)^3$	$\frac{\alpha\beta\gamma p a_1 b_1 c_1}{(S_1)^3}$
110	4	$\alpha\beta\gamma p a_1 b_1 c_0$	$(S_1)^2 \cdot (S_0)$	$\frac{\alpha\beta\gamma p a_1 b_1 c_0}{(S_1)^2 \cdot (S_0)}$
101	3	$\alpha\beta\gamma p a_1 b_0 c_1$	$(S_1)^2 \cdot (S_0)$	$\frac{\alpha\beta\gamma p a_1 b_0 c_1}{(S_1)^2 \cdot (S_0)}$
100	4	$\alpha\beta\gamma p a_1 b_0 c_0$	$(S_1) \cdot (S_0)^2$	$\frac{\alpha\beta\gamma p a_1 b_0 c_0}{(S_1) \cdot (S_0)^2}$
011	2	$\alpha\beta\gamma p a_0 b_1 c_1$	$(S_1)^2 \cdot (S_0)$	$\frac{\alpha\beta\gamma p a_0 b_1 c_1}{(S_1)^2 \cdot (S_0)}$
010	4	$\alpha\beta\gamma p a_0 b_1 c_0$	$(S_1) \cdot (S_0)^2$	$\frac{\alpha\beta\gamma p a_0 b_1 c_0}{(S_1) \cdot (S_0)^2}$
001	3	$\alpha\beta\gamma p a_0 b_0 c_1$	$(S_1) \cdot (S_0)^2$	$\frac{\alpha\beta\gamma p a_0 b_0 c_1}{(S_1) \cdot (S_0)^2}$
000	4	$\alpha\beta\gamma p a_0 b_0 c_0$	$(S_0)^3$	$\frac{\alpha\beta\gamma p a_0 b_0 c_0}{(S_0)^3}$

The parameters are estimated according to the previously introduced logic by determining first γ and then c_1 and c_0.

5 Applying the concept

5.1 Simulating population distribution scenarios

The previously introduced concept looks of course rather theoretical. Let us first emphasize that according to fractal logic the localizaton of the square-like elements can be modified within a certain range without affecting the fractal properties. Indeed we may chose the localization of the squares for the first step as we like if respecting the already enounced two rules:

- the squares must lie within the initial square;
- the squares are not allowed to overlap.

Hence we can e.g. put the big square in a corner of the initial square and place the four smaller ones around. In the next step the same logic holds. This means that *within* each square generated at the previous step, we can again place the now generated smaller squares by respecting the two introduced rules.

So the fractal rules just forbids that already generated lacunae i.e. conserved "free" space are not "occupied" by squares in further iteration steps[4].

The software allows identifying the population living in the introduced square-like elements. These empirical data can be used to estimate the above introduced model parameter, what means that we have for each scale (iteration step) the amounts α, $\beta \ldots$ of population living in the rural hinterland i.e. the zones making no longer part of the zones to be developed in future. Moreover are computed the means of the centers belonging to the same code according to the previously introduced logic, i.e. by a stepwise estimation of a_1, a_0, b_1, b_0 etc.

Moreover, the user can define himself values for these parameters in order to increase or reduce population concentration in certain types of centers which are, of course again distinguished according to their code. The software tool provides the information about these "theoretical" population numbers to the user, too and allows representing again the deficits or surpluses with respect to reality. Hence different kinds of scenarios can be constructed and compared. Similar simulation actually developed for the metropolitan area of Vienna (cf. [CYF11], [Cze12]).

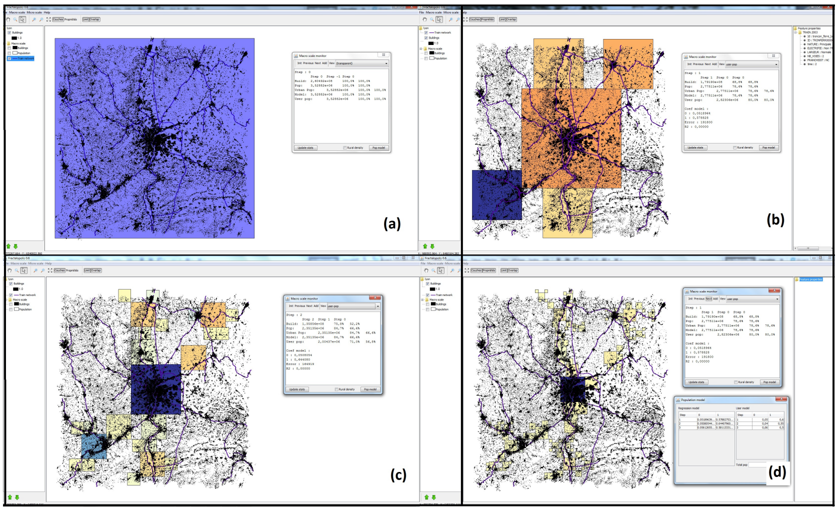


Figure 6: Application of the model on the urban area of Lyon: (a) shows the chosen area with the train network, (b) represents the first iteration step, (c) the second one and (d) the third one - on the bottem at the right the waiting factors for poppulation distribution ribtion used at the different steps

Figure 6 illustrates how we may apply the concept in a real world case, the area of the Lyon agglomeration. Lyon the second important metropolitan area in France. Lying at the western border of the Rhone valley, it benefits from a good accessibility to Mediterranean area as well as to Italy, but also to the Northern and Eastern part of France and to central Europe. This strategic position favored urban development since the antiquity. In the 19th century textile and silk industry boosted the urban development. Moreover heavy industry rose up at the same time in the South-West in the Gier valley around Sainte Etienne.

[4]Let us remind that in our case the lacunae may contain settlements, but this areas are not opened to future urbanization

Indeed the first railway line in France linked both these cities. Even the decline of the old industrial activities did not slow down the economic success and Lyon still remains a highly attractive region. Urbanization is constrained in the West by a mountain region, which is however attractive for high standard residential areas. In the East and the South of Lyon, the flat Rhone valley was propitious for setting up industrial areas and constructing social housing and lower level residential areas, whereas a swampy nowadays protected area prevented the North of strong urbanization. Several axes with subcenters can be identified :

- in the North the Saône Valley with Villefranche-sur-Saône and farther away Macon,
- in the North-East the well-developped axe in direction of Bourg-en-Bresse,
- in the South the Rhone valley with Vienne,
- in the South-West the Gier valley with Saint Etienne,
- in the South-East an axe going in direction Chambéry.

These axes are served by railway lines and are well accessible by road axes. In our simulation we considerd the four firstly enounced axes as propitious for further development and we localized the subcenters on existing important cities and areas favoured for further urbanization follows mainly the local railway lines[5]. Hence enlarging settlement will be restricted to already settled-up areas with good accessibility to the main center. Of course other axes could have been chosen or even the fifth axis could have been included by chosing another genertor consisting of 5 subcenters. In the example we have generated three iteration steps (figures (b) to (d)). The figure represents the situation where we have chosen some parameter values and computed the differences between real world situation and the population numbers computed by means of the model parameters. In the figures 6 (b) - (d) dark blue corresponds to an important surplus of population whith respect to the chosen parameter values, whereas red would indicate a high lack of population. Pale yellow corresponds to a rather good concordance between the real world situation and the model. Here we have illustrated a potential "redistribution" of population and we see that according to the iteration steps and the chosen parameters some surplus of population may occur at a certain step and disappear in the next one. This is due to the fact that population is really distributed among all the areas according to the model parameters and this may, in a next step, generate a spatial dsitribution being in better concordance with real world situation. We could also have added population and estimated how this surplus would have to been dispatched among the different centers.

The chosen parameters concentrate the population by using the following parameters (cf. figure 6(d) at the bottom at right):

$\alpha a_1 = 0.6$	$\alpha a_0 = 0.05$	$(1 - \alpha) = 0.2$
$\beta b_1 = 0.55$	$\beta b_0 = 0.04$	$(1 - \alpha) = 0.29$
$\gamma b_1 = 0.5$	$\gamma b_0 = 0.05$	$(1 - \gamma) = 0.30$

[5]The crossing high speed railway lines are not of interest in the given context.

This corresponds to a scenario where the concentration in the center is the highest on the global level and periurbanization is the more accepted the more the level is local. The rural hinterland is favored on the second and third level with respect to the first one, since $(1-\gamma) > (1-\beta) > (1-\alpha)$, what corresponds to the idea that a slight concentration close to development zones seems more reasonable than completely uniform distribution in rural areas.

5.2 Outlook

For concrete applications it is useful to combine the population model with data sets about the localization of the different kinds of services and facilities according to the defined levels. Then it is possible to define rules which allow evaluating for a given square-like element the quality of accessibility to the existing services, but also to the leisure areas of different attendance rate. For the basic model referring to a unifractal approach and a fixed grid covering the study area such rule sets are presented in detail in [TVHP12]). For the here discussed fractalopolis approach similar rule sets have been defined for the, adapted to the case of the Vienna agglomeration, which are presented in a companion paper which will be soon available ([FC12]).

Of course “virtual” services can be introduced if it is e.g. intended to establish somewhere a shopping mole or other types of facilities. By varying the position of the squares their best position for future urbanization can be tested.

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